

Invariance and Monotonicity for Stochastic Delay Differential Equations

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Abstract

We study invariance and monotonicity properties of Kunita-type stochastic differential equations in \mathbb{R}^d with delay. Our first result provides sufficient conditions for the invariance of closed subsets of \mathbb{R}^d . Then we present a comparison principle and show that under appropriate conditions the stochastic delay system considered generates a monotone (order-preserving) random dynamical system. Several applications are considered.

Keywords: stochastic delay/functional differential equation, stochastic flow, random dynamical system, invariance, monotonicity, random attractor.

1 Introduction

In this paper, we study invariance and monotonicity properties of a class of stochastic functional differential equations (sfde's) driven by a *Kunita-type* martingale field. Our main results are Theorem 3.4 on deterministic invariant domains and the comparison principle stated in Theorem 4.2. To prove them we represent the sfde as a random fde (see [22] and the references therein). From the point of view of deterministic delay systems this random fde has a non-standard structure and therefore we cannot apply the results on monotonicity available in the deterministic theory. This is why we are forced to develop a new method starting from the basic monotonicity ideas. We restrict our attention to a class of sfde's which generate a stochastic semi-flow on the state space of continuous functions (for an example of an sfde which does not generate such a semi-flow, see [19]). For other classes (on L_p -type spaces, for instance) we can use a variety of approximation procedures to achieve similar results. Our choice of continuous functions as a phase space is mainly motivated by the fact

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that some important results in the theory of monotone systems require a phase space with a solid minihedral cone (see, e.g., [14, 15]).

We note that invariance properties for deterministic functional differential equations have been discussed by many authors (see, e.g., [27, 17, 18] and the references therein). We also refer to [28] and to the literature quoted there for monotonicity properties of deterministic fde's. Stochastic and random ode's were considered in [4]. Similar questions for nonlinear stochastic partial differential equations (spde's) were studied in [3, 8] (see also [6, 7] and the references therein for other applications of monotonicity methods in spde's).

The paper is organized as follows.

In Section 2 we introduce basic definitions and hypotheses and describe the structure of our stochastic fde model in (2) and its random representation (see (5)). The central result in this section is Proposition 2.2 which shows the equivalence of the stochastic fde (2) and the random fde (5).

In Section 3 we establish our main result concerning the invariance of deterministic domains (see Theorem 3.4). The proof involves the random representation established in Proposition 2.2 and also the deterministic approach developed in [27]. As an application of Theorem 3.4 we consider an invariant regular simplex for stochastic delayed Lotka-Volterra type model.

In Section 4 we consider quasi-monotone vector (drift) fields and using the same idea as in Section 3 establish in Theorem 4.2 a comparison principle for the corresponding sfde's.

In Section 5 we apply the results of Sections 3 and 4 to construct *random dynamical systems* (RDS's) defined on invariant regions and generated by sfde's from the class considered (see Theorem 5.3). These RDS's become order-preserving for quasi-monotone drift fields (see Theorem 5.7). In this section following [1] (for the monotone case, see also [4]) we recall well-known notions of the theory of random dynamical systems including that of a *pull-back attractor*. Theorem 5.7 on the generation of a monotone RDS allows us apply results from the theory of monotone RDS's (see, e.g., [2, 4, 5] and the literature cited in these publications) to describe the qualitative dynamics of the sfde's considered. We discuss this issue briefly and provide several examples.

2 Preliminaries

Let $r > 0$, d a positive integer and let $C := C([-r, 0], \mathbb{R}^d)$ be the Banach space of continuous \mathbb{R}^d -valued functions equipped with the supremum norm $\|\cdot\|_C$. For a continuous \mathbb{R}^d -valued function x defined on some subset of \mathbb{R} containing the interval $[s - r, s]$, we define $x_s \in C$ by

$$x_s(u) := x(s + u), \quad u \in [-r, 0].$$

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions. On this probability space we define real-valued random fields M^i and G^i , $i = 1, 2, \dots, d$ satisfying the following hypotheses.

Hypothesis (M). For each $i = 1, 2, \dots, d$, $M^i : [0, \infty) \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$ satisfies

- (i) M^i is continuous in the first two variables for each $\omega \in \Omega$.
- (ii) For each $x \in \mathbb{R}^d$, $M^i(\cdot, x)$ is a local martingale and $M^i(0, x, \omega) = 0$ for all $\omega \in \Omega$.
- (iii) There exist $\delta \in (0, 1)$ and predictable processes $a^{ij} : [0, \infty) \times \mathbb{R}^{2d} \times \Omega \rightarrow \mathbb{R}$ such that for each $i, j \in \{1, \dots, d\}$:

$$R^{ij}(t, \omega) := \sup_{x, y \in \mathbb{R}^d} \frac{|a^{ij}(t, x, y)|}{(1 + |x|)(1 + |y|)} + \sup_{x, y \in \mathbb{R}^d} \|D_x D_y a^{ij}(t, x, y)\| \\ + \sup_{x \neq x', y \neq y'} \frac{\|\bar{a}^{ij}(t, x, y) - \bar{a}^{ij}(t, x, y') - \bar{a}^{ij}(t, x', y) + \bar{a}^{ij}(t, x', y')\|}{|x - x'|^\delta |y - y'|^\delta}$$

is finite, where $\bar{a}^{ij}(t, x', y') := D_x D_y a^{ij}(t, x', y')$ and

$$\langle M^i(\cdot, x), M^j(\cdot, y) \rangle_t = \int_0^t a^{ij}(s, x, y, \omega) ds \quad \text{a.s., } i, j = 1, \dots, d,$$

where $\langle M^i, M^j \rangle_t$ denotes the corresponding joint quadratic variation (see [16] for details). Moreover, we assume that the map $t \mapsto R^{ij}(t, \omega)$ is locally integrable w.r.t. Lebesgue measure for every $\omega \in \Omega$ and $i, j \in \{1, \dots, d\}$. In the definition of R^{ij} , $D_x D_y a$ denotes the matrix formed by the corresponding partial derivatives and $\|\cdot\|$ is an arbitrary norm on the space of matrices.

Hypothesis (G). $G = (G^1, \dots, G^d) : [0, \infty) \times C \times \Omega \rightarrow \mathbb{R}^d$ satisfies

- (i) G^i is jointly continuous in the first two variables for each $\omega \in \Omega$.
- (ii) For each $\omega \in \Omega$, bounded set B in C and $T > 0$ there exists some $L = L(T, B, \omega) < \infty$ such that $|G^i(t, \eta, \omega) - G^i(t, \zeta, \omega)| \leq L \|\eta - \zeta\|_C$ for all $0 \leq t \leq T$ and $\eta, \zeta \in B$.
- (iii) For each $\eta \in C$ and $t \in [0, \infty)$, $G(t, \eta)$ is \mathcal{F}_t -measurable.

Below, it will be important to decompose G as

$$G(t, \eta, \omega) = H(t, \eta, \omega) + b(t, \eta(0), \omega),$$

where both H and b satisfy (i), (ii) and (iii) of the previous hypothesis (with C replaced by \mathbb{R}^d with the Euclidean norm for b). In addition, we assume that $b(t, \cdot)$ is continuously differentiable for each t and ω and there exist $\delta > 0$ and a number $c(T, \omega) < \infty$ such that

$$\sup_{0 \leq t \leq T} \left\{ \sup_{x \in \mathbb{R}^d} \|Db(t, x, \omega)\| + \sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{\|Db(t, x, \omega) - Db(t, y, \omega)\|}{|x - y|^\delta} \right\} \leq c(T, \omega). \quad (1)$$

In this case we say that Hypothesis **(G)** holds with decomposition $G = H + b$.

For the rest of this section, we assume both hypotheses **(M)** and **(G)** and fix a particular decomposition $G = H + b$ as above.

We consider the following Kunita-type delay stochastic differential equation

$$\begin{cases} dx^i(t) &= G^i(t, x_t) dt + M^i(dt, x(t)), \quad i = 1, 2, \dots, d, \quad t \geq s, \\ x_s &= \eta, \end{cases} \quad (2)$$

where $s \geq 0$ and η is a C -valued \mathcal{F}_s -measurable random variable.

For the definition of *Kunita-type stochastic integrals*

$$\int_s^t M^i(du, x(u)),$$

for adapted and continuous (or more general) processes x , the reader is referred to Kunita's monograph [16]. Readers who are unwilling to learn Kunita integrals (even though they are very natural and easy to deal with objects) can think of the special case

$$M^i(t, x) := \sum_{k=1}^m \int_0^t \sigma^{ik}(s, x) dW^k(s), \quad (3)$$

where W^k , $k = 1, \dots, m$, are independent Brownian motions and the σ^{ik} are (deterministic) functions (satisfying appropriate regularity properties). In this case (2) reads

$$\begin{cases} dx^i(t) &= G^i(t, x_t) dt + \sum_{k=1}^m \sigma^{ik}(t, x(t)) dW^k(t), \quad i = 1, \dots, d, \quad t \geq s, \\ x_s &= \eta, \end{cases}$$

and $a^{ij}(t, x, y) = \sum_{k=1}^m \sigma^{ik}(t, x) \sigma^{jk}(t, y)$ is deterministic.

We aim at a representation of the solution from which one can read off continuity properties with respect to the initial condition. Note that even though equation (2) is easily seen to have a unique solution for each fixed s and η , continuity with respect to η does not follow since solutions are defined only up to a set of measure zero which may depend on η . To obtain continuity, one has to select a particular modification of the solution. We will use a variant of the *variation-of-constants* technique which turns (2) into an equation which does not contain any stochastic integral and can therefore be solved for each fixed $\omega \in \Omega$. We will see that the modification of the solution which is given by the pathwise equation does automatically exhibit continuous dependence upon the initial condition. The variation-of-constants technique, which is well-known for ode's, has already been applied to sfde's in [20] and [22].

For further use we need some properties of the following (non-delay) stochastic equation

$$\begin{cases} d\psi^i(t) &= b^i(t, \psi(t)) dt + M^i(dt, \psi(t)), \quad i = 1, 2, \dots, d, \quad t \geq s, \\ \psi^i(s) &= x, \end{cases} \quad (4)$$

where $\psi = (\psi^1, \dots, \psi^d)$. The following lemma states that equation (4) generates a stochastic flow of diffeomorphisms in \mathbb{R}^d . This is a special case of Theorem 4.6.5 in [16].

Lemma 2.1 *We assume that $b \equiv (b^1, \dots, b^d) : [0, \infty) \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ is a vector field satisfying (1). Then there exists a process $\Psi : [0, \infty)^2 \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ which satisfies the following:*

- (i) *For each $s \geq 0$ and $x \in \mathbb{R}^d$, $\psi(t) \equiv \Psi_{s,t}(x, \omega)$, $t \geq s$ solves equation (4).*
- (ii) *For each $s \geq 0$, $x \in \mathbb{R}^d$ and $\omega \in \Omega$, $\Psi_{s,s}(x, \omega) = x$.*
- (iii) *The maps $(s, t, x) \mapsto \Psi_{s,t}(x, \omega)$ and $(s, t, x) \mapsto D_x \Psi_{s,t}(x, \omega)$ are continuous for each $\omega \in \Omega$. Furthermore $\Psi_{s,t}(\cdot, \omega)$ is a C^1 -diffeomorphism for each $s, t \in \mathbb{R}$ and $\omega \in \Omega$.*
- (iv) *For each $s, t, u \geq 0$, and $\omega \in \Omega$ we have the semi-flow property*

$$\Psi_{s,u}(\cdot, \omega) = \Psi_{t,u}(\cdot, \omega) \circ \Psi_{s,t}(\cdot, \omega).$$

Note that by (ii) and (iv), we have $\Psi_{s,t}(\cdot, \omega) = (\Psi_{t,s}(\cdot, \omega))^{-1}$.

Lemma 2.1 allows us to construct the following representation for solutions to (2). In the special case in which the martingale field M is given by a finite number of Brownian motions as in (3) and $b \equiv 0$, this representation was established in Lemma 2.3 in [22].

Let $\Psi(u, x, \omega) := \Psi_{0,u}(x, \omega)$. We define the functions $\xi : [0, \infty) \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ and $F : [0, \infty) \times \mathbb{R}^d \times C \times \Omega \rightarrow \mathbb{R}$ by

$$\begin{aligned} \xi(u, x, \omega) &:= \Psi(u, \cdot, \omega)^{-1}(x) = \Psi_{u,0}(x, \omega), \\ F(u, x, \eta, \omega) &:= \{D_x \Psi(u, x, \omega)\}^{-1} H(u, \eta, \omega) \end{aligned}$$

and consider the (random) equation

$$x(t, \omega) = \Psi\left(t, \left[\xi(s, \eta(0), \omega) + \int_s^t F(u, \xi(u, x(u, \omega), \omega), x_u(\omega), \omega) du\right], \omega\right) \quad (5)$$

for $t \geq s$ with the initial data

$$x(t, \omega) = \eta(t - s) \quad \text{for } t \in [s - r, s], \quad (6)$$

where η is a C -valued \mathcal{F}_s -measurable random variable. We suppress the dependence of $x = (x^1, \dots, x^d)$ on s and η for notational simplicity. The following proposition shows that equations (2) and (5) (together with (6)) are equivalent.

Proposition 2.2 *Fix $s \geq 0$, a C -valued \mathcal{F}_s -measurable random variable η and a stopping time $T \geq s$. An adapted \mathbb{R}^d -valued process $x(t)$ with continuous paths solves equation (2) on the interval $[s, T(\omega)] \cap [s, \infty)$ with initial condition $x_s = \eta$ if and only if x satisfies (5) and (6) on the same interval for almost all $\omega \in \Omega$.*

Proof. The proof is essentially the same as that of Lemma 2.3 in [22]. Our assumptions are slightly different from the ones in [22] but this does not affect the arguments in the proof. Therefore, we skip some details.

First assume that x solves (5) and (6) on $[s, T(\omega)] \cap [s, \infty)$ for almost all $\omega \in \Omega$. Then $x_s = \eta$ almost surely. Equation (5), together with a slight modification of the generalized Itô's formula as stated in [16], Theorems 3.3.1 and 3.3.3(i) imply, that x is a continuous semimartingale and satisfies

$$\begin{aligned} dx(t) = & D_x \Psi \left(t, [\xi(s, \eta(0), \omega) + \int_s^t F(u, \xi(u, x(u, \omega), \omega), x_u(\omega), \omega) du], \omega \right) \\ & \times F(t, \xi(t, x(t), \omega), x_t, \omega) dt \\ & + \Psi(dt, [\xi(s, \eta(0), \omega) + \int_s^t F(u, \xi(u, x(u, \omega), \omega), x_u(\omega), \omega) du], \omega) \\ = & H(t, x_t) dt + b(t, x(t)) dt + M(dt, x(t)). \end{aligned}$$

Therefore x solves (2).

Conversely, suppose that x solves (2) and define

$$\zeta(t, \omega) := \xi(s, \eta(0), \omega) + \int_s^t F(u, \xi(u, x(u), \omega), x_u, \omega) du.$$

Let

$$\tilde{x}(t, \omega) \equiv (\tilde{x}^1(t, \omega), \dots, \tilde{x}^d(t, \omega)) := \begin{cases} \Psi(t, \zeta(t, \omega), \omega), & t \geq s \\ \eta(t - s), & t \in [s - r, s]. \end{cases}$$

One can see that $\tilde{x}^i(t, \omega)$ is a semimartingale with differential

$$d\tilde{x}^i(t) = H^i(t, x_t) dt + b^i(t, \tilde{x}(t)) dt + M^i(dt, \tilde{x}(t)), \quad i = 1, \dots, d.$$

This (non-retarded) sde has a unique solution \tilde{x} with initial condition $\tilde{x}(s) = \eta(0)$, so x and \tilde{x} agree on $[s - r, T] \cap [s, \infty)$ almost surely. This proves the proposition. \square

The following proposition provides a well-posedness result concerning problem (5) and (6).

Proposition 2.3 *Let hypotheses **(M)** and **(G)** be satisfied. Then there exists a set Ω_0 of full measure such that for all $\omega \in \Omega_0$, $s \geq 0$, and $\eta \in C$, the problem (5) and (6) has a unique local solution $x(s, \eta, t, \omega)$ up to an explosion time $\tau(s, \eta, \omega)$. For each $s \geq 0$, the solution depends continuously upon (t, η) (up to explosion). Further, the following semi-flow property holds: for all $0 \leq s \leq t \leq u$, all $\eta \in C$ and all $\omega \in \Omega_0$, we have*

$$x(s, \eta, u, \omega) = x(t, x_t(s, \eta, t, \omega), u, \omega) \quad \text{up to explosion.}$$

Proof. This is (essentially) Theorem 2.1 in [22]. The only differences are the fact that in [22] the authors use the Hilbert space M_2 instead of C as the state

space and that we separate b from G and combine it with the martingale part. The proof of local existence, uniqueness and continuity of the problem (5) and (6) is based on a rather standard fixed point argument. \square

It is natural to ask for sufficient conditions for the explosion time $\tau(s, \eta, \omega)$ to be infinite on a set of full measure which does not depend on s and η . We will say that condition **(GE)** (for *global existence*) holds if **(G)** and **(M)** hold with decomposition $G = H + b$ and there exists a set Ω_0 of full measure such that $\tau(s, \eta, \omega) = \infty$ for all $s \geq 0$, all $\eta \in C$ and all $\omega \in \Omega_0$. Various sufficient conditions for **(GE)** are formulated in Theorem 3.1 in [22]. They are based on spatial estimates on the growth of the flow Ψ and its spatial derivative which were established in [21] and [12]. We quote them here:

Proposition 2.4 *Let **(G)** and **(M)** hold with decomposition $G = H + b$. Each of the following conditions is sufficient for **(GE)**:*

- (i) *For each $T > 0$ and $\omega \in \Omega$ there exist $c = c(T, \omega)$ and $\gamma = \gamma(T, \omega) \in [0, 1)$ such that*

$$|H(t, \eta, \omega)| \leq c(1 + \|\eta\|_C^\gamma) \quad (7)$$

for all $0 \leq t \leq T$, $\eta \in C$ and $\omega \in \Omega$.

- (ii) *For each $T > 0$ there exists $\beta \in (0, r)$ such that $H(u, \eta, \omega) = H(u, \tilde{\eta}, \omega)$ holds for all $\omega \in \Omega$ whenever $0 \leq u \leq T$ and $\eta|_{[-r, -\beta]} = \tilde{\eta}|_{[-r, -\beta]}$.*

- (iii) *For all $\omega \in \Omega$ and $T \in (0, \infty)$ we have that*

$$\sup_{0 \leq u \leq T, x \in \mathbb{R}^d} \|(\mathbf{D}_x \psi(u, x, \omega))^{-1}\| < \infty$$

and there exists $c = c(T, \omega)$ such that (7) holds with $\gamma = 1$.

It is a bit annoying that (i) excludes the case of H satisfying a global Lipschitz condition. It seems to be open whether **(GE)** holds in that case.

3 Deterministic invariant regions

In this section we assume that Hypotheses **(M)** and **(G)** with decomposition $G = H + b$ and condition (7) are in force and consider a general problem of the form (5),(6). We provide sufficient conditions that, given a non-empty closed (deterministic) subset \mathbb{D} in \mathbb{R}^d , a solution with values in this set for $t \in [t_0 - r, t_0]$ will have values in \mathbb{D} for all $t > t_0$. The key idea is to decompose the solution semi-flow in such a way that Ψ alone leaves \mathbb{D} invariant and that the remaining drift does not change this property.

Below we use the notation

$$C_{\mathbb{D}} = \{\eta \in C : \eta(s) \in \mathbb{D} \text{ for every } s \in [-r, 0]\}. \quad (8)$$

We need some additional hypotheses (which are inspired by similar hypotheses for deterministic fde's in [27]).

Hypothesis (G_ε). There exists a family {G_ε} of random fields satisfying (G) with decomposition $G_\varepsilon = H_\varepsilon + b$ for every $\varepsilon \in (0, \varepsilon_0]$ such that

- (i) H_ε satisfies condition (7);
- (ii) $\lim_{\varepsilon \rightarrow 0} H_\varepsilon(t, \eta, \omega) = H(t, \eta, \omega)$ for every $(t, \eta) \in [0, \infty) \times C_\mathbb{D}$ and $\omega \in \Omega$;
- (iii) given $(t, \eta) \in [0, \infty) \times C_\mathbb{D}$ and $\varepsilon \in (0, \varepsilon_0]$ there exists an $\alpha = \alpha(\varepsilon, t, \eta, \omega) > 0$ such that if $0 < h \leq \alpha$ and $u \in \mathbb{R}^d$ is such that $|u| \leq \alpha$, then

$$\eta(0) + hH_\varepsilon(t, \eta, \omega) + hu \in \mathbb{D};$$

- (iv) if $y^\varepsilon(t, \eta)$ solves the problem (5),(6) with G_ε instead of G , then for every $\omega \in \Omega$, $t \geq 0$, and $\eta \in C$ we have $\lim_{\varepsilon \rightarrow 0} y^\varepsilon(t, \eta, \omega) = x(t, \eta, \omega)$, where $x(t, \eta, \omega)$ solves (5),(6).

Note that condition (ii) implies that \mathbb{D} is the closure of its interior. Further note that condition (iv) above is not implied by the other conditions – not even in the case of a deterministic ode, see [27].

Hypothesis (M_ℙ). The problem (4) generates a stochastic flow $\Psi_{t,s}(\cdot, \omega)$ of diffeomorphisms of \mathbb{R}^d such that

$$\Psi_{s,t}(\mathbb{D}, \omega) = \mathbb{D}, \quad t > s, \quad \omega \in \Omega. \quad (9)$$

Remark 3.1 For flows which are driven by a finite number of Brownian motions, explicit criteria for the validity of this hypothesis are well-known (we will state some of them below). We have not found corresponding criteria for Kunita-type equations in the mathematical literature. In fact, such criteria follow easily in case \mathbb{D} is compact: for Kunita-type sde's, the one-point motion (i.e. the solution for a single starting point $x \in \mathbb{R}^d$) can be described by an equivalent sde which is driven by a finite number of Brownian motions (which depend on the point x). Assuming that for each $x \in \mathbb{D}$ the solution stays in \mathbb{D} forever with probability one (for which one can check the known criteria), then the same holds true for a countable dense set of initial conditions in \mathbb{D} . The fact that \mathbb{D} is compact and the flow is continuous shows that there exists a set Ω_0 of full measure such that (9) holds for all $\omega \in \Omega$. Our claim follows since we are free to modify Ψ on a set of measure zero.

Remark 3.2 If \mathbb{D} is a closed convex subset of \mathbb{R}^d with nonempty interior then Hypothesis (G_ε) follows from the Nagumo type relation

$$\lim_{h \rightarrow 0+} h^{-1} \text{dist}(\eta(0) + hH(t, \eta, \omega), \mathbb{D}) = 0 \quad (10)$$

for any $t \in \mathbb{R}$, $\omega \in \Omega$ and $\eta \in C$ such that $\eta(s) \in \mathbb{D}$ for $s \in [-r, 0]$. In this case we can take

$$H_\varepsilon(t, \eta, \omega) = H(t, \eta, \omega) - \varepsilon(\eta(0) - \mathbf{e}),$$

where \mathbf{e} is an element from $\text{int } \mathbb{D}$. If $\mathbb{D} = \mathbb{R}_+^d$ relation (10) is equivalent to the requirement

$$\{\eta \geq 0, \eta^i(0) = 0\} \Rightarrow H^i(t, \eta, \omega) \geq 0 \quad (11)$$

for every $t, \omega \in \Omega$ and $i = 1, \dots, d$. For the proofs we refer to [27].

In the following remark we discuss conditions and examples when Hypothesis $(\mathbf{M}_{\mathbb{D}})$ is valid.

Remark 3.3 Assume that \mathbb{D} is a closed set in \mathbb{R}^d such that \mathbb{D} has an outer normal at every point of its boundary. We recall that a unit vector ν is said to be an *outer normal* to \mathbb{D} at the point $x_0 \in \partial\mathbb{D}$, if there exists a ball $B(x_1)$ with center at x_1 such that $B(x_1) \cap \mathbb{D} = \{x_0\}$ and $\nu = \lambda \cdot (x_1 - x_0)$ for some positive λ .

Let W_1, \dots, W_l be independent standard Wiener processes. We consider (4) with

$$M^i(dt, \psi(t)) = \sum_{j=1}^l m_j^i(\psi(t)) dW_j(t), \quad (12)$$

where the coefficients have bounded derivatives up to second order. The problem in (4) can be written as a Stratonovich sde:

$$\begin{cases} d\psi^i(t) &= \tilde{b}^i(t, \psi(t)) dt + \sum_{j=1}^l m_j^i(\psi(t)) \circ dW_j(t), \quad i = 1, \dots, d, \\ \psi^i(s) &= x, \end{cases}$$

where “ \circ ” denotes Stratonovich integration and

$$\tilde{b}^i(t, x) \equiv b^i(t, x) - \frac{1}{2} \sum_{j=1}^l \sum_{k=1}^d m_j^k(x) \frac{\partial m_j^i(x)}{\partial x_k}.$$

It follows from Wong-Zakai type arguments that \mathbb{D} is forward invariant under ψ if for any $x \in \partial\mathbb{D}$ we have

$$\sum_{i=1}^d \tilde{b}^i(t, x) \nu_x^i \leq 0 \quad \text{and} \quad \sum_{i=1}^d m_j^i(x) \nu_x^i = 0, \quad j = 1, \dots, l, \quad (13)$$

for every outer normal $\nu_x = (\nu_x^1, \dots, \nu_x^d)$ to \mathbb{D} at x . We refer to [4, Chap.2, Corollaries 2.5.1 and 2.5.2] for details. Further, $(\mathbf{M}_{\mathbb{D}})$ holds if \mathbb{D} is both forward and backward invariant under ψ . For this to hold it is sufficient to assume that the first inequality in (13) is an equality for each $x \in \partial\mathbb{D}$. We note that in the case $\mathbb{D} = \mathbb{R}_+^d$ and $b^i(t, x) \equiv 0$ the first condition in (13) follows from the second one which can be written in the form

$$m_j^i(x) = 0 \quad \text{for all } x = (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_d), \quad i = 1, \dots, d, \quad j = 1, \dots, l.$$

As an example we point out the case when $\mathbb{D} = \{(x_1; x_2) : x_1^2 + x_2^2 \leq 1\} \subset \mathbb{R}^2$ and problem (4) has the form

$$dx_i = m^i(x_1, x_2)dW(t), \quad i = 1, 2.$$

In this case $b^i(t, x) \equiv 0$ and relations (13) holds if $m^1(x) = m^2(x) = 0$ for all $|x| = 1$. For instance, we can take $m^1(x_1, x_2) = -c(|x|)x_2$ and $m^2(x_1, x_2) = c(|x|)x_1$, where $c(r) = 0$ for $r = 1$.

Our main result in this section is the following theorem.

Theorem 3.4 *Assume that Hypotheses (\mathbf{G}) , (\mathbf{M}) , $(\mathbf{M}_{\mathbb{D}})$ and $(\mathbf{G}_{\varepsilon})$ hold. Then \mathbb{D} is a forward invariant set for problem (2) in the sense that for any $\omega \in \Omega$, $s \geq 0$ and $\eta \in C$ such that $\eta(u) \in \mathbb{D}$ for $u \in [-r, 0]$, the (unique) solution x of (5) and (6) satisfies $x(t, \eta, \omega) \in \mathbb{D}$ for all $t \geq s$.*

Proof. Assume that \mathbb{D} is not forward invariant. Then there exist $\omega \in \Omega$, $s \geq 0$, $\eta \in C_{\mathbb{D}}$ and $t_* > s$ such that $x(t_*) \notin \mathbb{D}$. Let $y^\varepsilon(t)$ be a solution to the auxiliary problem in $(\mathbf{G}_{\varepsilon})(iv)$. It follows from assumption (iv) in $(\mathbf{G}_{\varepsilon})$ that there exist $\varepsilon > 0$ and $t_0 \in [s, t_*)$ such that

$$y^\varepsilon(t) \in \mathbb{D}, \quad t \in [s - r, t_0] \quad \text{and} \quad y^\varepsilon(t_0 + h_j) \notin \mathbb{D},$$

where $\{h_j\}$ is a sequence of positive numbers such that $\lim_{j \rightarrow \infty} h_j = 0$. The solution $y^\varepsilon(t)$ can be represented in the form

$$y^\varepsilon(t, \omega) = \psi(t, \zeta^\varepsilon(t, \omega), \omega), \quad t \geq t_0,$$

where

$$\zeta^\varepsilon(t, \omega) = \xi(t_0, y^\varepsilon(t_0), \omega) + \int_{t_0}^t F_\varepsilon(u, \xi(u, y^\varepsilon(u), \omega), y_u^\varepsilon, \omega) du.$$

Here $\xi(t, x, \omega) = \psi(t, \cdot, \omega)^{-1}x$, $\psi(t, \cdot, \omega) = \Psi_{0,t}(\cdot, \omega)$, where $\Psi_{0,t}(\cdot, \omega)$ is the diffeomorphism given by (4) and

$$F_\varepsilon(u, x, \eta, \omega) = \{D_x \psi(u, x, \omega)\}^{-1} H_\varepsilon(u, \eta, \omega).$$

Since $(u, x, \eta) \mapsto F_\varepsilon(u, x, \eta, \omega)$ is continuous, we have that

$$\int_{t_0}^{t_0+h} F_\varepsilon(u, \xi(u, y^\varepsilon(u), \omega), y_u^\varepsilon, \omega) du = h F_\varepsilon(t_0, \xi(t_0, y^\varepsilon(t_0), \omega), y_{t_0}^\varepsilon, \omega) + o(h).$$

Thus

$$\zeta^\varepsilon(t_0 + h, \omega) = \xi(t_0, y^\varepsilon(t_0), \omega) + h F_\varepsilon(t_0, \xi(t_0, y^\varepsilon(t_0), \omega), y_{t_0}^\varepsilon, \omega) + o(h).$$

We have that $\psi(t_0, \psi^{-1}(t_0, x, \omega), \omega) = x$. Therefore by the chain rule

$$D_z \psi(t_0, \psi^{-1}(t_0, x, \omega), \omega) D_x \psi^{-1}(t_0, x, \omega) = \text{Id}.$$

Thus

$$\{D_z \psi(t_0, \psi^{-1}(t_0, x, \omega), \omega)\}^{-1} = D_x \psi^{-1}(t_0, x, \omega).$$

Consequently,

$$F_\varepsilon(t_0, \xi(t_0, y^\varepsilon(t_0), \omega), y_{t_0}^\varepsilon, \omega) = D_x \psi^{-1}(t_0, y^\varepsilon(t_0), \omega) H_\varepsilon(t_0, y_{t_0}^\varepsilon, \omega).$$

It is also clear that

$$\begin{aligned} \psi^{-1}(t_0, y^\varepsilon(t_0) + h H_\varepsilon(t_0, y_{t_0}^\varepsilon, \omega), \omega) - \psi^{-1}(t_0, y^\varepsilon(t_0), \omega) \\ = h D_x \psi^{-1}(t_0, y^\varepsilon(t_0), \omega) H_\varepsilon(t_0, y_{t_0}^\varepsilon, \omega) + o(h). \end{aligned}$$

Thus

$$\zeta^\varepsilon(t_0 + h, \omega) = \psi^{-1}(t_0, y^\varepsilon(t_0) + h H_\varepsilon(t_0, y_{t_0}^\varepsilon, \omega), \omega) + o(h).$$

This implies that

$$\psi(t_0, \zeta^\varepsilon(t_0 + h, \omega), \omega) = y^\varepsilon(t_0) + h H_\varepsilon(t_0, y_{t_0}^\varepsilon, \omega) + o(h). \quad (14)$$

Hypothesis (\mathbf{G}_ε) (iii) implies that the right-hand side of (14) is in \mathbb{D} for all sufficiently small $h > 0$. By Hypothesis $(\mathbf{M}_\mathbb{D})$ we therefore have $\zeta^\varepsilon(t_0 + h, \omega) \in \mathbb{D}$ and hence $y^\varepsilon(t_0 + h, \omega) = \psi(t_0 + h, \zeta^\varepsilon(t_0 + h, \omega), \omega) \in \mathbb{D}$ for all sufficiently small $h > 0$ contradicting our assumption that $y^\varepsilon(t_0 + h_j, \omega) \notin \mathbb{D}$ for all j . This contradiction proves the theorem. \square

In the following assertion we show that, similarly to the deterministic situation (see [28] and the references therein), in some cases the Nagumo type condition in (10) provides us necessary and sufficient conditions for invariance.

Corollary 3.5 *Let \mathbb{D} be a closed convex subset of \mathbb{R}^d with nonempty interior and let Hypotheses (\mathbf{G}) , (\mathbf{M}) and $(\mathbf{M}_\mathbb{D})$ be in force. Then \mathbb{D} is a forward invariant set if and only if (10) holds.*

Proof. If (10) holds, then we can apply Remark 3.2 to conclude that \mathbb{D} is forward invariant.

Let \mathbb{D} be forward invariant. It is clear that (14) holds for $\varepsilon = 0$ and $t_0 = s$, i.e. we have

$$\psi(s, \zeta(s + h, \omega), \omega) = \eta(0) + h H(s, \eta, \omega) + o(h) \quad (15)$$

for any $\eta \in C_\mathbb{D}$. Since \mathbb{D} is invariant, we have $\psi(s + h, \zeta(s + h, \omega), \omega) = x(s + h, \eta, \omega) \in \mathbb{D}$ for all $h \geq 0$. $(\mathbf{M}_\mathbb{D})$ implies that $\psi(s, \zeta(s + h, \omega), \omega)$ lies in \mathbb{D} for all $h \geq 0$. Therefore (15) implies (10). \square

In the case $\mathbb{D} = \mathbb{R}_+^d$ Corollary 3.5 implies the following assertion.

Corollary 3.6 *Let (\mathbf{G}) and (\mathbf{M}) be in force with M^i of the form (12). Assume that*

$$b^i(t, x) = 0, \quad m_j^i(x) = 0 \quad \forall x = (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_d),$$

where $i = 1, \dots, d$, $j = 1, \dots, l$. Then \mathbb{R}_+^d is forward invariant set if and only if (11) holds.

Proof. It follows from Corollary 3.5, see also Remarks 3.3 and 3.2. \square

More complicated example of an invariant set \mathbb{D} is discussed in the following remark.

Remark 3.7 Assume that \mathbb{D} is a set of the form

$$\mathbb{D} = \{x \in \mathbb{R}^d : \langle a_q, x \rangle \leq \gamma_q, \quad q = 1, \dots, Q\},$$

where $a_q = (a_q^1, \dots, a_q^d) \in \mathbb{R}^d$, $\gamma_q \in \mathbb{R}$, $q = 1, \dots, Q$. Then (10) holds if and only if for every $q = 1, \dots, Q$ we have the relation

$$\sum_{i=1}^d a_q^i H^i(t, \eta, \omega) \leq 0$$

whenever $\eta \in C_{\mathbb{D}}$ and $\langle a_q, \eta(0) \rangle = \gamma_q$.

In the case when $b^i \equiv 0$ and M^i has the form (12), it follows from Remark 3.3 that Condition $(\mathbf{M}_{\mathbb{D}})$ holds if for every $q = 1, \dots, Q$ we have the relations

$$\sum_{j=1}^l \sum_{k=1}^d \sum_{i=1}^d m_j^k(x) \frac{\partial m_j^i(x)}{\partial x_k} a_q^i = 0, \quad \sum_{i=1}^d m_j^i(x) a_q^i = 0, \quad j = 1, \dots, l,$$

whenever $\langle a_q, x \rangle = \gamma_q$. For instance, this condition is true if

$$m_j^i(x) = \sigma_j^i(x_1, \dots, x_d) \cdot \prod_{q=1}^Q h_q(\langle a_q, x \rangle - \gamma_q),$$

where $\sigma_j^i(x_1, \dots, x_d)$ are arbitrary and $h_q(s)$ is such that $h_q(0) = 0$.

Now we provide some examples.

Example 3.8 For the system

$$dx^i(t) = x^i(t) f^i(x_t) dt + x^i(t) \sum_{j=1}^l \sigma_j^i(x^1(t), \dots, x^d(t)) dW_j(t), \quad i = 1, \dots, d,$$

the set \mathbb{R}_+^d is a forward invariant set. Here f^i and σ_j^i are such that condition (\mathbf{GE}) holds. This conclusion follows from Corollary 3.6.

Example 3.9 In the previous example, the noise can also be replaced by a more general *Kunita-type* noise. As a particular example, let N be *space-time white noise* on $\mathbb{R}^d \times [0, \infty)$, let $h : \mathbb{R}^d \rightarrow [0, \infty)$ be C^∞ with compact support and define

$$M^i(dt, x) := \phi(x^i) \int_{\mathbb{R}^d} h(x - z) N(dz, dt),$$

where $\phi \in C^\infty$ is bounded and all its derivatives are bounded. Assume that $\phi(0) = 0$. Then

$$\langle M^i(., x), M^j(., y) \rangle_t = \phi(x^i) \phi(y^j) t \int_{\mathbb{R}^d} h(x-z) h(y-z) dz.$$

Note that M satisfies hypothesis **(M)** due to our assumptions on ϕ and h . To facilitate things, we assume that $f^i(\eta) = g^i(\eta(-r))$ where g^i is bounded and Lipschitz. Then the set \mathbb{R}_+^d is a forward invariant set for the flow generated by

$$dx^i(t) = \phi(x^i(t)) g^i(x(t-r)) dt + M^i(dt, x(t)), \quad i = 1, \dots, d.$$

To see this, first note that condition **(GE)** holds by Proposition 2.4(i). Then we argue as in Remark 3.1: for each given starting point $x \in \mathbb{R}_+^d$, the solution starting in x will remain in \mathbb{R}_+^d forever almost surely since it can be written as a solution to an sfde with finitely many driving Brownian motions, so the same property holds for all starting points in \mathbb{R}_+^d with rational coordinates. Since we know that a local flow exists, no trajectory of the flow can leave \mathbb{R}_+^d through one of the hyperplanes bordering \mathbb{R}_+^d . Since also **(GE)** holds, no trajectory of the flow can escape to infinity at finite time either, so the set \mathbb{R}_+^d is invariant.

Note that in this set-up the driving noise M is independent at locations x and y with distance larger than the diameter of the support of h which is a reasonable assumption in many models and which cannot be achieved with a finite number of driving Wiener processes.

Example 3.10 (Lotka-Volterra type model) Consider the system

$$dx^i(t) = -\alpha_i x^i(t) (1 - \langle b, x(t-r) \rangle) dt + \sigma_i x^i(t) (1 - \langle b, x(t) \rangle) dW_i(t), \quad i = 1, \dots, d, \quad (16)$$

where $b \in \mathbb{R}_+^d$, $\alpha_i \geq 0$, $\sigma_i \in \mathbb{R}$. The set $\mathbb{D} = \{x \in \mathbb{R}_+^d : \langle b, x \rangle \leq 1\}$ is forward invariant. Since \mathbb{D} is a bounded set in \mathbb{R}^d , we can modify the nonlinear terms outside some vicinity of \mathbb{D} in order to satisfy the requirement in (7). This allows us to apply Propositions 2.3 and 2.4(i) and obtain well-posedness of the problem in (16). The statement on the invariance follows from Theorem 3.4 via the observation made in Remark 3.7.

4 Comparison theorem for sfde's

Our next result is a comparison principle for functional differential equations perturbed by Kunita type noise of the form (2) with the local martingales M^i not only satisfying Hypothesis **(M)** but also

$$M^i(t, x, \omega) = M^i(t, x_i, \omega) \quad \text{for all } i = 1, \dots, d, \quad x = (x_1, \dots, x_d),$$

i.e., M^i depends on t , ω and on the i -th component of spatial variable x only. As can be seen from [4], this structural requirement is needed for a comparison principle even in the non-delay case.

Thus, instead of (2), we consider the following Kunita-type retarded stochastic differential equation

$$\begin{cases} dx^i(t) &= G^i(t, x_t) dt + M^i(dt, x^i(t)), \quad i = 1, 2, \dots, d, \quad t \geq s, \\ x_s &= \eta, \end{cases} \quad (17)$$

where η is a C -valued \mathcal{F}_s -measurable random variable and the drift terms G^i satisfies **(G)**. We fix a decomposition $G = H + b$ as in the previous section and assume that b^i depends on x^i only. In this case instead of (4) we have the diagonal system of scalar non-delay equations

$$\begin{cases} d\psi^i(t) &= b^i(t, \psi^i(t))dt + M^i(dt, \psi^i(t)), \quad i = 1, 2, \dots, d, \quad t \geq s, \\ \psi^i(s) &= x \in \mathbb{R}. \end{cases} \quad (18)$$

It follows from Lemma 2.1 that equation (18) generates a stochastic flow of diffeomorphisms $x \mapsto \psi_{s,t}^i(x, \omega)$ in \mathbb{R} for each $i = 1, 2, \dots, d$. Moreover,

$$\Psi_{s,t}(x, \omega) = (\psi_{s,t}^1(x_1, \omega), \dots, \psi_{s,t}^d(x_d, \omega))$$

satisfies all statements of Lemma 2.1. Below we often write $\psi^i(t, x, \omega)$ instead of $\psi_{0,t}^i(x, \omega)$. Observe that due to the diffeomorphic property the flow ψ^i is automatically *strongly monotone* in the sense that for $x, y \in \mathbb{R}$ we have

$$x < y \text{ implies } \psi_{s,t}^i(x, \omega) < \psi_{s,t}^i(y, \omega) \text{ for each } s, t \geq 0, \omega \in \Omega.$$

Indeed, if the implication above is not true, then there exist $x < y$, $s < t$, and ω such that $\psi_{s,t}^i(x, \omega) = \psi_{s,t}^i(y, \omega)$. Since $\psi_{s,t}^i(\cdot, \omega)$ is invertible, this implies $x = y$ and thus provides a contradiction.

Applying Proposition 2.2 we can specify representations (5) and (6) for our case of diagonal M^i . Namely, if we define

$$\begin{aligned} \xi^i(u, x^i, \omega) &:= \psi^i(u, \cdot, \omega)^{-1}(x^i) \\ F^i(u, x^i, \eta, \omega) &:= \{D_{x^i} \psi^i(u, x^i, \omega)\}^{-1} H^i(u, \eta, \omega), \end{aligned}$$

then (5) can be written in the form

$$x^i(t, \omega) = \psi^i \left(t, \left[\xi^i(s, \eta^i(0), \omega) + \int_s^t F^i(u, \xi^i(u, x^i(u, \omega), \omega), x_u(\omega), \omega) du \right], \omega \right) \quad (19)$$

for all $t \geq s$.

Let C_+ be the standard cone in C . This cone defines a partial order relation via

$$\eta \geq \eta_* \text{ iff } \eta - \eta_* \in C_+, \quad (20)$$

i.e., iff $\eta^i(s) \geq \eta_*^i(s)$ for all $s \in [-r, 0]$ and $i = 1, \dots, d$, where

$$\eta = (\eta^1, \dots, \eta^d) \quad \text{and} \quad \eta_* = (\eta_*^1, \dots, \eta_*^d)$$

are elements from $C = C([-r, 0], \mathbb{R}^d)$. We write $\eta > \eta_*$ iff $\eta \geq \eta_*$ and $\eta \neq \eta_*$ and use the notation $\eta >> \eta_*$ if

$$\eta^i(s) > \eta_*^i(s) \text{ for all } s \in [-r, 0] \text{ and } i = 1, \dots, d.$$

We also consider another sfde

$$\begin{cases} dx^i(t) &= \bar{G}^i(t, x_t) dt + M^i(dt, x^i(t)), \quad i = 1, 2, \dots, d, \quad t \geq s, \\ x_s &= \eta_* \in C, \end{cases} \quad (21)$$

with the same M and b . We assume that the random field $\bar{G} = \{\bar{G}^i\}$ satisfies Hypothesis **(G)** with decomposition $\bar{G} = \bar{H} + b$. Let $\mathbb{D}(\omega) \subseteq \mathbb{R}^d$ be a closed set with nonempty interior.

Definition 4.1 Let $\mathbb{D}(\omega) \subseteq \mathbb{R}^d$ be a closed set with nonempty interior. For each ω , let $[a(\omega), b(\omega)]$ a random interval in \mathbb{R}_+ . A random vector field $G = (G^1, \dots, G^d) : [0, \infty) \times C \times \Omega \rightarrow \mathbb{R}^d$ is said to be *quasimonotone* on $[a(\omega), b(\omega)] \times \mathbb{D} \subseteq \mathbb{R}^{d+1}$ iff for any $\eta = (\eta^1(s), \dots, \eta^d(s))$ and $\eta_* = (\eta_*^1(s), \dots, \eta_*^d(s))$ from $C_{\mathbb{D}}$, where $C_{\mathbb{D}}$ is defined by (8), we have the following implication

$$\{\eta \geq \eta_*, \eta^i(0) = \eta_*^i(0)\} \Rightarrow G^i(t, \eta, \omega) \geq G^i(t, \eta_*, \omega)$$

for every $t \in [a(\omega), b(\omega)]$, $\omega \in \Omega$ and $i = 1, \dots, d$.

We note for future use that quasimonotonicity is invariant with respect to a decomposition $G = H + b$ with b^i depending on x^i only in the sense that G is quasimonotone if and only if H is quasimonotone.

Theorem 4.2 (Comparison Principle) *Assume that M satisfies the conditions above and that the random vector fields $G = (G^1, \dots, G^d)$ and $\bar{G} = (\bar{G}^1, \dots, \bar{G}^d)$ satisfy Hypothesis **(G)**. Let $x(t) := x(t, \eta, \omega)$ be a solution to (17) and $y(t) = y(t, \eta, \omega)$ be a solution to (21) which possess the property*

$$x(t), y(t) \in \mathbb{D}, \quad \text{for } t \in [s, s + T(\omega)]$$

*for some convex closed set $\mathbb{D} \subseteq \mathbb{R}^d$ with nonempty interior, where $T(\omega) > 0$ for all $\omega \in \Omega$. Assume that **(M_D)** holds and the random field G is quasimonotone on $[s, s + T(\omega)] \times \mathbb{D}$. Then the following assertions hold:*

1. *If $\eta \leq \eta_*$ and*

$$G(t, \xi, \omega) \leq \bar{G}(t, \xi, \omega) \text{ for all } \xi \in C_{\mathbb{D}}, t \in [s, s + T(\omega)], \omega \in \Omega, \quad (22)$$

then

$$x(t; \eta, \omega) \leq y(t, \eta_*, \omega) \text{ for all } t \in [s, s + T(\omega)], \omega \in \Omega. \quad (23)$$

2. *If $\eta \geq \eta_*$ and*

$$G(t, \xi, \omega) \geq \bar{G}(t, \xi, \omega) \text{ for all } \xi \in C_{\mathbb{D}}, t \in [s, s + T(\omega)], \omega \in \Omega, \quad (24)$$

then

$$x(t; \eta, \omega) \geq y(t, \eta_*, \omega) \text{ for all } t \in [s, s + T(\omega)], \omega \in \Omega. \quad (25)$$

Proof. We prove the first part only (the proof of the reversed inequalities is similar).

We start with the case $\eta << \eta_*$ and $G << \bar{G}$, i.e., we assume that

$$\eta^i(s) < \eta_*^i(s) \text{ for all } s \in [-r, 0] \text{ and } i = 1, \dots, d, \quad (26)$$

and

$$G^i(t, \xi, \omega) < \bar{G}^i(t, \xi, \omega) \text{ for all } \xi \in C_{\mathbb{D}}, t \in [s, s + T(\omega)], \omega \in \Omega, \quad (27)$$

where $i = 1, \dots, d$. The same is true for H and \bar{H} . Let us prove that

$$x^i(t; \eta, \omega) < y^i(t, \eta_*, \omega) \text{ for all } t \in [s, s + T(\omega)], \omega \in \Omega, i = 1, \dots, d. \quad (28)$$

Since $x(t)$ and $y(t)$ are continuous for all $\omega \in \Omega$ relation (28) is valid for some interval $[s, s + \tau(\omega)]$, where $0 < \tau(\omega) \leq T(\omega)$. If (28) does not hold for *all* t from $[s, s + T(\omega)]$, then for some ω there exist $t' \in (0, T(\omega))$ and $i \in \{1, \dots, d\}$ such that

$$x^i(t') = y^i(t') \text{ and } x^j(t) < y^j(t) \text{ for all } t \in [s, s + t'], j = 1, \dots, d.$$

Using representation (19) and strict monotonicity of ψ^j we obtain that

$$\zeta_G^i(t') = \zeta_G^i(t') \text{ and } \zeta_G^j(t) < \zeta_G^j(t) \text{ for all } t \in [s, s + t'], j = 1, \dots, d, \quad (29)$$

where

$$\begin{aligned} \zeta_G^j(t) &:= \xi^j(s, \eta^j(0), \omega) + \int_s^t F_G^j(u, \xi^j(u, x^j(u), \omega), x_u, \omega) du, \\ \zeta_{\bar{G}}^j(t) &:= \xi^j(s, \eta_*^j(0), \omega) + \int_s^t F_{\bar{G}}^j(u, \xi^j(u, y^j(u), \omega), y_u, \omega) du \end{aligned}$$

with the following notation:

$$\begin{aligned} \xi^j(u, x^j, \omega) &:= \psi^j(u, \cdot, \omega)^{-1}(x^j), \\ F_G^j(u, x^j, \eta, \omega) &:= \{D_{x^j} \psi^j(u, x^j, \omega)\}^{-1} H^j(u, \eta, \omega), \\ F_{\bar{G}}^j(u, x^j, \eta, \omega) &:= \{D_{x^j} \psi^j(u, x^j, \omega)\}^{-1} \bar{H}^j(u, \eta, \omega). \end{aligned}$$

Since the functions $F_G^j(u, x^j, \eta, \omega)$ and $F_{\bar{G}}^j(u, x^j, \eta, \omega)$ are continuous for every $\omega \in \Omega$, the processes $\zeta_G^j(t)$ and $\zeta_{\bar{G}}^j(t)$ are continuously differentiable and satisfy the equations

$$\frac{d}{dt} \zeta_G^j(t) = F_G^j(t, \xi^j(t, x^j(t), \omega), x_t, \omega) \quad (30)$$

and

$$\frac{d}{dt} \zeta_{\bar{G}}^j(t) = F_{\bar{G}}^j(t, \xi^j(t, y^j(t), \omega), y_t, \omega). \quad (31)$$

It follows from (29) that

$$\zeta_G^i(t') - \zeta_G^i(t) > \zeta_G^i(t') - \zeta_G^i(t) \quad \text{for all } s \leq t < s + t'.$$

This implies that

$$\frac{d}{dt}\zeta_G^i(t') \geq \frac{d}{dt}\zeta_G^i(t'). \quad (32)$$

However, since $x^i(t') = y^i(t')$, from (31) we have that

$$\frac{d}{dt}\zeta_G^i(t') = \{D_x \psi^j(u, x^i(t'), \omega)\}^{-1} H^i(t, y_{t'}, \omega).$$

Therefore (27) written for H and \bar{H} , quasimonotonicity of H and (30) imply that

$$\begin{aligned} \frac{d}{dt}\zeta_G^i(t') &> \{D_x \psi^j(u, x^i(t'), \omega)\}^{-1} H^i(t', y_{t'}, \omega) \\ &\geq \{D_x \psi^j(u, x^i(t'), \omega)\}^{-1} H^i(t', x_{t'}, \omega) = \frac{d}{dt}\zeta_G^i(t'). \end{aligned}$$

This relation contradicts to (32). Thus (26) and (27) imply (28).

To prove (23) for the general case we first apply the result above to the corresponding equations with

$$\bar{G}_\varepsilon(t, \eta, \omega) = \bar{G}(t, \eta, \omega) + \varepsilon(\mathbf{e}_1 - \eta(0))$$

and

$$G_\varepsilon(t, \eta, \omega) = G(t, \eta, \omega) + \varepsilon(\mathbf{e}_2 - \eta(0)),$$

where $\mathbf{e}_1, \mathbf{e}_2 \in \text{int } \mathbb{D}$ and $\mathbf{e}_1 << \mathbf{e}_2$. It is clear that (22) implies that

$$G_\varepsilon^i(t, \xi, \omega) < \bar{G}_\varepsilon^i(t, \xi, \omega) \quad \text{for all } \xi \in C_{\mathbb{D}}, t \in [s, s + T(\omega)], \omega \in \Omega, i = 1, \dots, d.$$

Thus, by limit transition we obtain (23) in the case when $\eta << \eta_*$. Using this fact it is easy to prove (23) for every $\eta \leq \eta_*$. \square

Remark 4.3 If the drift term G is quasimonotone on \mathbb{R}_+^d , then we have that $G(t, \eta, \omega) \geq G(t, 0, \omega)$ for every $\eta \in C_{\mathbb{R}_+^d}$. Therefore applying the comparison principle in (24) and (25) with $\bar{G} \equiv 0$ we can conclude that \mathbb{R}_+^d is a forward invariant set with respect to sfde (17) when $G(t, 0, \omega) \geq 0$ and $M(t, 0, \omega) \equiv 0$.

Example 4.4 (Lotka-Volterra type model) Consider the system

$$\left\{ \begin{aligned} dx^i(t) &= \alpha_i x^i(t) \left(1 - \beta_i x^i(t) - \sum_{j=1}^d c_{ij} \int_{-r}^0 x^j(t + \tau) d\mu_{ij}(\tau) \right) dt \\ &\quad + \sigma_i x^i(t) (R_i - x^i(t)) dW_i, \quad t > s, \quad i = 1, \dots, d, \\ x_s &= \eta \in C. \end{aligned} \right. \quad (33)$$

Here α_i, β_i and R_i are positive numbers, $c_{ij} \geq 0, \sigma_i \in \mathbb{R}$. We assume that $\mu_{ij}(\tau)$ are left continuous nondecreasing functions on $[-r, 0]$ of bounded variation such that

$$\mu_{ij}(0) - \mu_{ij}(-r) = 1, \quad i, j = 1, \dots, d.$$

It is easy to see from Theorem 3.4 (see also Corollary 3.6 and Remark 3.7) that $\mathbb{D} = \prod_{i=1}^d [0, R_i]$ is a forward invariant set for sfde (33) provided $R_i \geq \beta_i^{-1}$ for every $i = 1, \dots, d$.

We note that the global well-posedness of (33) follows from Propositions 2.3 and 2.4(ii) because we can modify the corresponding drift term outside \mathbb{D} to satisfy (7).

It is also clear that the functions

$$\bar{G}^i(\eta) := \alpha_i \eta^i(0) \left(1 - \beta_i \eta^i(0) - \sum_{j=1}^d c_{ij} \int_{-r}^0 \eta^j(\tau) d\mu_{ij}(\tau) \right), \quad i, j = 1, \dots, d,$$

satisfy the inequality

$$\alpha_i \eta^i(0) \left(1 - \beta_i \eta^i(0) - \sum_{j=1}^d c_{ij} R_j \right) \leq \bar{G}^i(\eta) \leq \alpha_i \eta^i(0) (1 - \beta_i \eta^i(0))$$

for every $\eta \in C_{\mathbb{D}}$, where $C_{\mathbb{D}}$ is given by (8) with $\mathbb{D} = \prod_{i=1}^d [0, R_i]$. Since the functions

$$G_1^i(\eta) := \alpha_i \eta^i(0) \left[1 - \beta_i \eta^i(0) - \sum_{j=1}^d c_{ij} R_j \right] \quad \text{and} \quad G_2^i(\eta) := \alpha_i \eta^i(0) (1 - \beta_i \eta^i(0))$$

are quasimonotone, Theorem 4.2 implies that for any initial data $\eta \in C_{\mathbb{D}}$ a solution

$$x(t, \eta, \omega) = (x^1(t, \eta, \omega), \dots, x^d(t, \eta, \omega))$$

to problem (33) satisfies the inequality

$$u^i(t, \eta, \omega) \leq x^i(t, \eta, \omega) \leq v^i(t, \eta, \omega), \quad i, j = 1, \dots, d, \quad (34)$$

where $u(t, \eta, \omega) = (u^1(t, \eta, \omega), \dots, u^d(t, \eta, \omega))$ solves the problem

$$\begin{cases} du^i(t) &= \alpha_i u^i(t) \left(1 - \beta_i u^i(t) - \sum_{j=1}^d c_{ij} R_j \right) dt \\ &\quad + \sigma_i u^i(t) (R_i - u^i(t)) dW_i, \quad t > s, \quad i = 1, \dots, d, \\ u^i(0) &= \min_{\tau \in [-r, 0]} \eta^i(s), \quad i = 1, \dots, d, \end{cases} \quad (35)$$

and $v(t, \eta, \omega) = (v^1(t, \eta, \omega), \dots, v^d(t, \eta, \omega))$ solves the problem

$$\begin{cases} dv^i(t) &= \alpha_i v^i(t) (1 - \beta_i v^i(t)) dt + \sigma_i v^i(t) (R_i - v^i(t)) dW_i, \quad t > s, \\ v^i(0) &= \max_{\tau \in [-r, 0]} \eta^i(s), \quad i = 1, \dots, d. \end{cases} \quad (36)$$

We emphasize that problems (35) and (36) are direct sums of one-dimensional ordinary stochastic differential equations. Long time dynamics of these 1D systems is described with details (see, e.g., [4] and the references therein). Thus we can use the relations in (34) to “localize” dynamics of the original sfde (33).

5 Order-preserving RDS generated by sfde’s

In this section we consider some other applications of Theorems 3.4 and 4.2 from point view of theory of random dynamical systems (RDS).

5.1 Generation of RDS in an invariant region

Following the monograph of Arnold [1], we introduce the notion of a random dynamical system.

Definition 5.1 Let X be a topological space. A *random dynamical system* (RDS) with time \mathbb{R}_+ and state space X is a pair (ϑ, ϕ) consisting of the following two objects:

1. A metric dynamical system (MDS) $\vartheta \equiv (\Omega, \mathcal{F}, \mathbf{P}, \{\vartheta(t), t \in \mathbb{R}\})$, i.e., a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a family of measure preserving transformations $\{\vartheta(t) : \Omega \mapsto \Omega, t \in \mathbb{R}\}$ such that
 - (a) $\vartheta(0) = \text{id}$, $\vartheta(t) \circ \vartheta(s) = \vartheta(t+s)$ for all $t, s \in \mathbb{R}$;
 - (b) the map $(t, \omega) \mapsto \vartheta(t)\omega$ is measurable and $\vartheta(t)\mathbf{P} = \mathbf{P}$ for all $t \in \mathbb{R}$.
2. A (perfect) cocycle ϕ over ϑ of continuous mappings of X with one-sided time \mathbb{R}_+ , i.e. a measurable mapping

$$\phi : \mathbb{R}_+ \times \Omega \times X \mapsto X, \quad (t, \omega, x) \mapsto \phi(t, \omega)x$$

such that (a) the mapping $\phi(\cdot, \omega) : x \mapsto \phi(t, \omega)x$ is continuous for all $t \geq 0$ and $\omega \in \Omega$; (b) it satisfies the cocycle property:

$$\phi(0, \omega) = \text{id}, \quad \phi(t+s, \omega) = \phi(t, \vartheta(s)\omega) \circ \phi(s, \omega)$$

for all $t, s \geq 0$ and $\omega \in \Omega$.

Definition 5.2 Let ϑ be an MDS, $\bar{\mathcal{F}}$ the \mathbf{P} -completion of \mathcal{F} and $\mathbf{F} = \{\mathcal{F}_t, t \in \mathbb{R}\}$ a family of sub- σ -algebras of $\bar{\mathcal{F}}$ such that

1. $\mathcal{F}_s \subseteq \mathcal{F}_t$, $s < t$;

2. $\mathcal{F}_s = \bigcap_{h>0} \mathcal{F}_{s+h}$, $s \in \mathbb{R}$, i.e. the filtration \mathbf{F} is right-continuous;
3. \mathcal{F}_s contains all sets in \mathcal{F} of \mathbf{P} -measure 0, $s \in \mathbb{R}$;
4. $\vartheta(s)$ is $(\mathcal{F}_{t+s}, \mathcal{F}_t)$ -measurable for all $s, t \in \mathbb{R}$.

Then (ϑ, \mathbf{F}) is called a *filtered metric dynamical system* (FMDS). If - in addition - (ϑ, ϕ) is an RDS such that $\phi(t, \cdot)x$ is $(\mathcal{F}_t, \mathcal{B}(X))$ -measurable for every $t \geq 0, x \in X$, then $(\vartheta, \mathbf{F}, \phi)$ is called a *filtered random dynamical system* (FRDS).

We recall that an X -valued stochastic process $Y(t), t \in T \subseteq \mathbb{R}$ is called *adapted* or *nonanticipating* with respect to the filtration \mathbf{F} if $Y(t)$ is $(\mathcal{F}_t, \mathcal{B}(X))$ -measurable for every $t \in T$. Therefore $(\vartheta, \mathbf{F}, \phi)$ is an FRDS iff (ϑ, ϕ) is an RDS, (ϑ, \mathbf{F}) is an FMDS and $\phi(\cdot, \cdot)x$ is adapted to \mathbf{F} for every $x \in X$.

Theorem 5.3 *Assume that Hypotheses $(\mathbf{M}_{\mathbb{D}})$ and $(\mathbf{G}_{\varepsilon})$ are in force and (\mathbf{GE}) (see Proposition 2.4) holds. If the drift term $G(t, \eta, \omega) \equiv G(\eta)$ and the local characteristic a of M are deterministic and autonomous, then problem (5),(6) (and hence (2)) generates a FRDS (ϑ, φ) in $C_{\mathbb{D}}$, where $C_{\mathbb{D}}$ is defined by (8) (the case $\mathbb{D} \equiv \mathbb{R}^d$ is not excluded). The corresponding cocycle φ has the form*

$$[\varphi(t, \omega)\eta](\tau) = \begin{cases} x(t + \tau, \eta, \omega), & t + \tau > 0, \\ \eta(t + \tau), & t + \tau \leq 0, \end{cases}$$

for every $\tau \in [-r, 0]$, where $x(t, \eta, \omega)$ is a solution to problem (2) with $s = 0$. Moreover $\varphi(t, \omega)$ is compact mapping in $C_{\mathbb{D}}$, i.e. for any bounded set A from $C_{\mathbb{D}}$, the set $\varphi(t, \omega)A$ is relatively compact in $C_{\mathbb{D}}$ for every $t > 0$.

Proof. It follows from Theorem 3.4 and from the representation in (5),(6) of solutions to (2). We also use Propositions 2.2, 2.3 and 2.4. \square

To describe long-time dynamics of an RDS we need a notion of a random set (see, e.g., [1] and the references therein).

Definition 5.4 A mapping $\omega \mapsto D(\omega)$ from Ω into the collection of all subsets of a separable Banach space V is said to be *random closed set*, iff $D(\omega)$ is a closed set for any $\omega \in \Omega$ and $\omega \mapsto \text{dist}_V(x, D(\omega))$ is measurable for any $x \in V$. The random closed set $D(\omega)$ is said to be *compact*, if $D(\omega)$ is compact for each ω . The random closed set $D(\omega)$ is said to be *tempered* if

$$D(\omega) \subset \{x \in V : \|x\|_V \leq r(\omega)\}, \quad \omega \in \Omega,$$

where the random variable $r(\omega)$ possesses the property $\sup_{t \in \mathbb{R}} \{r(\vartheta(t)\omega)e^{-\gamma|t|}\} < \infty$ for any $\gamma > 0$.

We also need the following concept of a random attractor of an RDS (see [9, 26] and also [1, 4] and the references therein). Below we denote by X any subset of separable Banach space V equipped with the induced topology.

Let \mathcal{D} be a family of random closed sets in X which is closed with respect to inclusions (i.e. if $D_1 \in \mathcal{D}$ and a random closed set $\{D_2(\omega)\}$ possesses the property $D_2(\omega) \subset D_1(\omega)$ for all $\omega \in \Omega$, then $D_2 \in \mathcal{D}$). Sometimes the collection \mathcal{D} is called a *universe* of sets (see [1]).

Definition 5.5 Suppose that (ϑ, φ) is an RDS in X . Let \mathcal{D} be a universe. A random closed set $\{A(\omega)\}$ from \mathcal{D} is said to be a *random pull-back attractor* of the RDS (ϑ, φ) in \mathcal{D} if $A(\omega) \neq X$ for every $\omega \in \Omega$ and the following properties hold:

- (i) A is an invariant set, i.e. $\varphi(t, \omega)A(\omega) = A(\vartheta(t)\omega)$ for $t \geq 0$ and $\omega \in \Omega$;
- (ii) A is attracting in \mathcal{D} , i.e. for all $D \in \mathcal{D}$

$$\lim_{t \rightarrow +\infty} d_X\{\varphi(t, \vartheta(-t)\omega)D(\vartheta(-t)\omega) \mid A(\omega)\} = 0, \quad \omega \in \Omega, \quad (37)$$

where $d_X\{A \mid B\} = \sup_{x \in A} \text{dist}_X(x, B)$.

If instead of (37) we have that

$$\lim_{t \rightarrow +\infty} \mathbf{P}\{\omega : d_X\{\varphi(t, \omega)D(\omega) \mid A(\vartheta(t)\omega)\} \geq \delta\} = 0$$

for any $\delta > 0$, then is said to be a *random weak attractor* of the RDS (ϑ, φ) .

Some authors (e.g. [9, 1]) require a random attractor to be compact (and do not insist that it is different from the whole space). This distinction will not be important in what follows. The notion of a *weak random attractor* was introduced in [23]. For the relation between weak, pull-back and forward attractors we refer to [25].

Remark 5.6 If \mathbb{D} is bounded in \mathbb{R}^d , then it is easy to see that the RDS (ϑ, φ) generated by (2) in $C_{\mathbb{D}}$ has a random compact pull-back attractor in the universe \mathcal{D} all bounded sets. Since by Theorem 5.3 the RDS (ϑ, φ) is compact, this follows from Theorem 1.8.1 [4], for instance. In the case of unbounded sets \mathbb{D} (e.g., $\mathbb{D} = \mathbb{R}_+^d$) we need some conditions which guarantee dissipativity of the corresponding RDS. These conditions can be obtained in the same way as for the non-delay case (see, for instance, [4, Theorem 6.5.1]).

5.2 Monotone RDS

Let as above $C = C([-r, 0], \mathbb{R}^d)$ and C_+ be the standard cone in C of nonnegative elements:

$$C_+ = \{\eta = (\eta^1(s), \dots, \eta^d(s)) \in C \mid \eta^i(s) \geq 0 \quad \forall s \in [-r, 0], \quad i = 1, \dots, d\}.$$

This cone is a normal solid minihedral cone. This fact is important for further application of the theory of monotone RDS. We refer to [14] and [15] for more details concerning cones and partially ordered spaces.

Let \mathbb{D} be a convex closed set in \mathbb{R}^d with nonempty interior. In the space $C_{\mathbb{D}}$ given by (8) we define a partial order relation via (20), i.e., $\eta \geq \eta_*$ iff $\eta - \eta_* \in C_+$.

Theorem 5.7 *Assume that the hypotheses of Theorem 5.3 hold. Let \mathbb{D} be a convex closed set in \mathbb{R}^d with nonempty interior and (ϑ, φ) be the FRDS generated by problem (17) in $C_{\mathbb{D}}$ defined by (8). If the random field G is quasimonotone in \mathbb{D} then (ϑ, φ) is an order-preserving FRDS in $C_{\mathbb{D}}$ which means that*

$$\eta \leq \xi \text{ in } C_{\mathbb{D}} \text{ implies } \varphi(t, \omega)\eta \leq \varphi(t, \omega)\xi \text{ for all } t \geq 0 \text{ and } \omega \in \Omega.$$

Proof. This follows from Theorem 4.2 with $\bar{G} \equiv G$. \square

Theorem 5.7 makes it possible to apply the general theory of monotone RDS (see [4] and also [2, 5]) to the class of sfde's considered. In particular it is possible to obtain the following results:

- To provide transparent conditions which guarantee the existence of stochastic equilibria and a compact pull-back attractor (see, e.g., the general Theorem 3.5.1 in [4]). We recall (see [1]) that a random variable $u : \Omega \mapsto C_{\mathbb{D}}$ is said to be an *equilibrium* (or fixed point, or stationary solution) of the RDS (ϑ, φ) if it is invariant under φ , i.e. if

$$\varphi(t, \omega)u(\omega) = u(\vartheta(t)\omega) \quad \text{a.s. for all } t \geq 0.$$

One can see that any equilibrium $u(\omega) \in C_{\mathbb{D}}$ for (ϑ, φ) has the form $u(\tau, \omega) = v(\vartheta(\tau)\omega)$, $\tau \in [-r, 0]$, where $v(\omega)$ is a random variable in \mathbb{D} .

- To describe the pull-back attractor for (ϑ, φ) as a compact set lying between two of its equilibria (see general Theorem 3.6.2 in [4]).
- In the case when there exists a probability measure π on the Borel σ -algebra of subsets $C_{\mathbb{D}}$, such that the law $\mathcal{L}(\varphi(t, \omega)x)$ weakly converges to π in $C_{\mathbb{D}}$, the system (ϑ, φ) has a random weak attractor $A(\omega)$ which is singleton, i.e. $A(\omega) = \{v(\omega)\}$, where the random variable $v(\omega) \in C_{\mathbb{D}}$ is an equilibrium (see the general Theorem 1 proved in [5]). We note that sufficient conditions for the existence of an invariant probability measure π and weak convergence of transition probabilities to π for an sfde (monotone or not) have been established for example in [13], [24], [10], and [11].

Example 5.8 Consider the stochastic equations

$$dx^i(t) = (g_0^i(x^1(t), \dots, x^d(t)) + g_1^i(x^1(t-r_1), \dots, x^d(t-r_d)))dt + m^i(x^i)dW_i \quad (38)$$

for $i = 1, \dots, d$. We assume that g_0^i , g_1^i and m^i are smooth functions which are globally Lipschitz. Under these conditions we can apply Proposition 2.4(iii) to guarantee global well-posedness for (38). Moreover, one can see from Theorem 5.7 that equations (38) generate an order-preserving RDS in the space $C = C([-r, 0]; \mathbb{R}^d)$ with $r = \max_i r_i$ provided that

$$\frac{\partial g_0^i(x)}{\partial x_i} \geq 0, \quad x \in \mathbb{R}^d, \quad i \neq j,$$

and $g_1^i(x)$ is monotone, i.e. for every $i = 1, \dots, d$ we have that

$$g_1^i(x^1, \dots, x^d) \leq g_1^i(y^1, \dots, y^d) \quad \text{when } x^j \leq y^j, \quad j = 1, \dots, d.$$

The following example is a special case of Example 5.8.

Example 5.9 Let W be standard Brownian motion. Consider 1D the retarded stochastic differential equation

$$dx(t) = (f(x(t)) + g(x(t-1)))dt + \sigma(x(t))dW(t),$$

where $f, g, \sigma : \mathbb{R} \rightarrow \mathbb{R}$ are Lipschitz, g is monotone and σ is strictly positive. By Theorem 5.7 this equation generates an order-preserving RDS in $C([-1, 0], \mathbb{R})$. Assume that the associated Markov semigroup on $C([-1, 0], \mathbb{R})$ admits an invariant (or *stationary*) measure (sufficient conditions are provided in [24]). In this case we can apply Theorem 1 [5] and conclude that the corresponding RDS has a unique equilibrium which is a weak random attractor.

It is known (cf. [29]) that in the case $\sigma \equiv 0$ the attractor of this system can contain multiple equilibria and also a periodic orbit. Thus we observe here that adding the noise term simplifies essentially long-time behaviour of the system (see also [5] for some details).

Example 5.10 (Stochastic biochemical control circuit) We consider the following system of Stratonovich stochastic equations

$$dx^1(t) = (g(L_d x_t^d) - \alpha_1 x^1(t))dt + \sigma_1 \cdot x^1(t) \circ dW_t^1, \quad (39)$$

$$dx^j(t) = (L_{j-1} x_t^{j-1} - \alpha_j x^j(t))dt + \sigma_j \cdot x^j(t) \circ dW_t^j, \quad j = 2, \dots, d. \quad (40)$$

Here as above “ \circ ” denotes Stratonovich integration, σ_j are nonnegative and α_j are positive constants, $j = 1, \dots, d$, and $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a C^1 function such that

$$0 < g(u) \leq au + b, \quad \text{and} \quad g'(u) \geq 0 \quad \text{for every } u > 0$$

for some constants a and b . We also use the notation $x_{j,t}(s) = x_j(t+s)$ for $s \in [-r_j, 0]$ and

$$L_j \eta = \int_{-r_j}^0 \eta(s) d\mu_j(s),$$

where $\mu_j : [-r_j, 0] \mapsto \mathbb{R}$ is nondecreasing, $\mu_j(-r_j) = 0$, $\mu_j(0) = 1$, $\mu_j(s) > 0$ for $s > -r_j$. We denote $r = \max_j r_j$ and equip (39) and (40) with initial data

$$x^i(t) = \xi^i(t) \geq 0, \quad t \in [-r, 0], \quad i = 1, \dots, d. \quad (41)$$

A deterministic version of this system was considered in [28], the stochastic non-retarded case was studied in [4], see also [5].

Let $\xi \in C([-r, 0] : \mathbb{R}_+^d)$ by Proposition 2.3 a local solution

$$x(t) = (x^1(t), \dots, x^d(t))$$

exists on some interval $[0, T(\omega))$. By Theorem 3.4 (see also Remark 3.2 and Corollary 3.6) we have that \mathbb{R}_+^d is a forward invariant set, i.e. $x(t) \in \mathbb{R}_+^d$ for all

$t \in [0, T(\omega))$. Applying Comparison Principle (see Theorem 4.2) we conclude that

$$0 \leq x(t) \leq \bar{x}(t) \quad \text{for all } t \in [0, T(\omega)), \quad (42)$$

where $\bar{x}^i(t) = (\bar{x}^1(t), \dots, \bar{x}^d(t))$ solves the following system of linear equations

$$dx^1(t) = (aL_d x_t^d - \alpha_1 x^1(t) + b)dt + \sigma_1 \cdot x^1(t) \circ dW_t^1, \quad (43)$$

$$dx^j(t) = (L_{j-1} x_t^{j-1} - \alpha_j x^j(t))dt + \sigma_j \cdot x^j(t) \circ dW_t^j, \quad j = 2, \dots, d. \quad (44)$$

with initial data (41). The structure of (43) and (44) allows us to solve these equations. Indeed, if we consider the drift part of the problem:

$$dx^j(t) = \sigma_j \cdot x^j(t) \circ dW_t^j, \quad x^j(0) = x, \quad j = 1, \dots, d.,$$

then $\psi^j(t, x) = x \exp\{\sigma_j W^j(t)\}$ solves it. The spatial derivative of $\psi^j(t, x)$ and its inverse are independent of x and thus we can apply Proposition 2.4(iii) to prove global existence of the solution \bar{x} . Due to (42) this implies that the solution $x(t)$ of (39), (40) and (41) does not explode at finite time and thus equations (39) and (40) generate an RDS (ϑ, φ) in $C_+ = C([-r, 0] : \mathbb{R}_+^d)$, where $r = \max_i r_i$. By Theorem 5.7 this RDS is order-preserving. By Comparison Theorem 4.2, this system is dominated from above by the affine RDS $(\vartheta, \varphi_{af})$ generated by (43) and (44).

Now we concentrate on the case $a = 0$ (this means that $g(u)$ is bounded). In this case we can construct an equilibrium $v(\omega) = (v^1(\omega), \dots, v^d(\omega))$ for $(\vartheta, \varphi_{af})$ by the formulas

$$v_1(\omega) = b \int_{-\infty}^0 e^{\alpha_1 t - \sigma_1 W_t^1} dt,$$

and

$$v_j(\omega) = \int_{-\infty}^0 L_{j-1} v_t^{j-1} \cdot e^{\alpha_j t - \sigma_j W_t^j} dt, \quad j = 2, \dots, d,$$

where $v_t^j(\omega) := v^j(\vartheta(t + \tau))$, $\tau \in [-r, 0]$. Since $\varphi(t, \omega)x \leq \varphi_{af}(t, \omega)x$ for every $x \in C_+$, it is easy to see that $v(\omega)$ is a super-equilibrium for (ϑ, φ) , i.e.,

$$\varphi(t, \omega)u(\omega) \leq v(\vartheta_t \omega) \quad \text{a.s. for all } t \geq 0.$$

Thus by Theorem 3.5.1 [4] the RDS (ϑ, φ) has an equilibrium $u(\omega) \in \mathbb{R}_+^d$. If $g(0) > 0$, this equilibrium is strongly positive.

We can also show that in the case $a = 0$ the RDS (ϑ, φ) possesses a random pull-back attractor in the universe of all tempered subsets of $C([-r, 0] : \mathbb{R}_+^d)$. Indeed, due to the compactness property of the cocycle ϑ (see Theorem 5.3) it is sufficient to prove that (ϑ, φ) possesses a bounded absorbing set. This set can be constructed in the following way.

Let $v(\omega)$ be the equilibrium for $(\vartheta, \varphi_{af})$ constructed above. One can see that in this case $v_\lambda(\omega) = \lambda v(\omega)$ is a super-equilibrium for RDS $(\vartheta, \varphi_{af})$ for every $\lambda > 1$. One can also see that the top Lyapunov exponent for $(\vartheta, \varphi_{af})$ with

$a = b = 0$ is negative. This implies $v_\lambda(\omega)$ is an *absorbing* super-equilibrium for $(\vartheta, \varphi_{af})$, i.e. for every tempered set $D(\omega)$ in $C([-r, 0] : \mathbb{R}_+^d)$ there is $t_D(\omega)$ that

$$\varphi_{af}(t, \vartheta_{-t}\omega)y(\vartheta_{-t}\omega) \leq v_\lambda(\omega), \quad t \geq t_D(\omega), \quad y \in D.$$

Since $(\vartheta, \varphi_{af})$ dominates (ϑ, φ) , this implies that the interval

$$[0, v_\lambda(\omega)] = \{u \in C : 0 \leq u \leq v_\lambda(\omega)\}$$

is absorbing for (ϑ, φ) . Therefore Theorem 1.8.1[4] implies the existence of a pullback attractor which belongs to some interval of the form $[u_1(\omega), u_2(\omega)]$, where $[u_1(\omega)$ and $u_2(\omega)]$ are two equilibria such that $0 \leq u_1(\omega) \leq u_2(\omega) \leq v(\omega)$.

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